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Structure of normal form series for non-analytical vector fields and generalized resonances

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Abstract. This paper is devoted to the study of normal form transformations and resonances. The usual theory of normal forms is formulated in a more general context: the quasi-monomial formalism, in which negative and non-integer exponents are accepted. The general coefficient of the Poincaré series is explicitly constructed in the non-resonant case, for any QM system. From there arises the necessity to generalize resonances to non-analytical vector fields. Using particular changes of parameterization, we extend this resonance relation to the nonlinear part of the vector field. The changes of variables that arise from this provide approximations of the solutions far from the fixed point.

1. Introduction

The Poincaré–Dulac normal form approach for solving nonlinear systems of ODEs is among the most interesting methods in computer algebra. However, the iteration of the existing algorithms [1] leads to extremely heavy calculations that cannot be performed by hand. Even using computer algebra languages, the construction of the Poincaré series appeared to be very difficult, since, among other problems, the general structure of the coefficient was still unknown. This question has been recently solved, independently by Ecalle [2], and by the present authors [3].

This paper is not only an extended version of [3]. It also contains new results: in section 4, using particular changes of time parametrization, we find criteria (the generalized resonance conditions) for the system to be conjugated to a system that is linear. The transformation leading to this simplified (integrable) system is close to the Poincaré transformation, and can thus also be explicitly constructed.

Hereafter, we introduce the usual notion of normal form and recall the principle of resonance. We also give a short introduction to the quasi-monomial (QM) notation that will be used throughout this paper.

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1.1. Normal forms

In a general context, normal form theory [4–6] applies to analytical systems of ODEs. Consider the following system:

$$\dot{x}_i = \lambda_i x_i + \sum_{\substack{\boldsymbol{m} \\ |\boldsymbol{m}| \geqslant 2}} a_i(\boldsymbol{m}) x^{\boldsymbol{m}} \qquad (i = 1, \ldots, N) \tag{1}$$

where the dot denotes the derivative with respect to the independent variable (say the time t for instance), and where we used a multi-index notation:

$$\boldsymbol{m} = (m_1, m_2, \ldots, m_N)$$
$$x^{\boldsymbol{m}} = x_1^{m_1} x_2^{m_2} \ldots x_N^{m_N}$$
$$\text{and} \quad |\boldsymbol{m}| = m_1 + m_2 + \cdots + m_N.$$

The m_i are positive integer numbers (since the system is analytic), and the origin is a fixed point. The condition $|\boldsymbol{m}| \geqslant 2$ ensures that $\sum a_i(\boldsymbol{m}) x^{\boldsymbol{m}}$ is purely nonlinear.

The principle of normal form theory is to find an analytical change of coordinates, with the origin as a fixed point, such that the vector field becomes simpler to study in terms of the new variables, i.e. we try to find variables for which the system is linear, or at least, to remove some inessential part of the nonlinearity. Doing so the system of nonlinear ODEs is reduced to a system of linear PDEs: the homological equations for the transformation. If we find the linearizing change of variables, it means that the phase portrait around the origin is topologically equivalent to the phase portrait of a linear system; in other words: the nonlinearity does not affect the qualitative behaviour of the system.

Practically, let us expand the change of variables in Taylor series:

$$x_i = y_i + \sum_{\substack{\boldsymbol{m} \\ |\boldsymbol{m}|>1}} b_i(\boldsymbol{m}) y^{\boldsymbol{m}}$$

where we assume analyticity of the mapping and its proximity to the identity. Using this in (1), gives a new system

$$\dot{y}_i = \lambda_i y_i + \sum_{\substack{\boldsymbol{m} \\ |\boldsymbol{m}|>1}} c_i(\boldsymbol{m}) y^{\boldsymbol{m}} \tag{2}$$

where the linear part is unchanged. Now let us try to fix the coefficients $b_i(\boldsymbol{m})$ in order to remove monomials in this new system. Doing this, we have to solve, order by order, the hierarchy of algebraic equations:

$$(\boldsymbol{m} \cdot \boldsymbol{\lambda} - \lambda_i) b_i(\boldsymbol{m}) = F_i(\boldsymbol{m}) \tag{3}$$

where $(\boldsymbol{m} \cdot \boldsymbol{\lambda})$ stands for $\sum_{k=1}^{N} m_k \lambda_k$, and the $F_i(\boldsymbol{m})$ are some functions of the λ_j, and of the $a_i(\boldsymbol{p})$ and $b_i(\boldsymbol{p})$ (with $p_i \leqslant m_i$, and $|\boldsymbol{p}| < |\boldsymbol{m}|$). In general, the recursion (3) cannot be solved explicitly in compact form: it has to be studied order by order. It can at least be formally solved provided that there is no vanishing factor $(\boldsymbol{m} \cdot \boldsymbol{\lambda} - \lambda_i)$; otherwise, in general, the corresponding equation cannot be solved. The relation

$$\boldsymbol{m} \cdot \boldsymbol{\lambda} - \lambda_i = 0$$

is called a *resonance condition*. When a resonance condition is satisfied, the new system still contains nonlinear monomials; these are called *resonant monomials*. In the non-resonant case, the change of variable is also an expression of the solution as a series of exponentials of the independent variable t, since the new equations are simply linear. It can happen that some $(\boldsymbol{m} \cdot \boldsymbol{\lambda} - \lambda_i)$ and simultaneously the corresponding $F_i(\boldsymbol{m})$ are vanishing. In this case, the

coefficients $b_i(\boldsymbol{m})$ are free (and are usually chosen equal to zero). Thus, the best we can do is to find new coordinates for which the coefficients of (2) satisfy

$$c_i(\boldsymbol{m}) = 0 \qquad \text{if} \quad \boldsymbol{m}\cdot\boldsymbol{\lambda} - \lambda_i \neq 0.$$

The system (2) is then called the *normal form* of the system (1). The transformation leading to it is the *Poincaré transformation*. The normal form system can be reduced further [6, 7] (i.e. some resonant monomials can be removed) by a judicious choice of the coefficients $b_i(\boldsymbol{m})$ of the resonant monomials in the Poincaré series. However, the system will still contain nonlinear terms.

An important question is: when is the Poincaré series convergent? From the relation (3), we see that

$$b_i(\boldsymbol{m}) \sim (\boldsymbol{m}\cdot\boldsymbol{\lambda} - \lambda_i)^{-1}$$

and, even if no resonance condition is satisfied, the quantity $(\boldsymbol{m}\cdot\boldsymbol{\lambda} - \lambda_i)$ may become close to zero as $\boldsymbol{m}$ increases; the corresponding $b_i(\boldsymbol{m})$ will become large, and this can cause the divergence of the Poincaré series.

All we can do is to find conditions on the spectrum $\boldsymbol{\lambda}$, such that the Poincaré series is convergent, no matter what the nonlinearity is. These conditions are thus sufficient but not necessary conditions. The less restrictive known condition is Bruno's diophantine condition [6] (equivalent to a condition found by Rüssman [16] for diffeomorphisms): let us define

$$\omega(k) = \inf(|\boldsymbol{m}\cdot\boldsymbol{\lambda} - \lambda_i|) \qquad \text{with} \quad k = |\boldsymbol{m}|.$$

Then the Poincaré series is convergent if

$$\sum_{k\geqslant 1} \frac{1}{2^k} \log\left(\frac{1}{\omega(2^k)}\right) < \infty.$$

Note that the convergence domain of the Poincaré series is a neighbourhood of the origin in phase space, i.e. a certain domain around zero for the variables y_i. This means that the convergence depends on the initial conditions $y_i(0)$, that are themselves functions of the initial conditions $x_i(0)$ (via the inverse of the Poincaré transformation). The choice of a set of initial conditions $[x_i(0)]$ selects a trajectory in phase space. The Poincaré series converges during the time interval for which this trajectory crosses the neighbourhood of the origin. Hence, this time interval may not exist if the series diverges.

The convergence theorem can also be applied to resonant systems: the coefficients for the resonant monomials are then left free, and only the non-vanishing $(\boldsymbol{m}\cdot\boldsymbol{\lambda} - \lambda_i)$) are taken into account in the definition of $\omega(k)$.

1.2. The QM formalism

The QM formalism [8–10] characterizes the system by two real or complex rectangular constant matrices. Their dimension depends on the nonlinearity of the system, and can be infinite.

QM differential systems are systems that can be written as

$$\dot{x}_i = \alpha_i x_i + x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{N} x_k^{B_{jk}} \qquad (i = 1, \ldots, N) \tag{4}$$

where m is arbitrary. The A_{ij} and the B_{jk} are thus real or complex constants; this class is quite general, since (4) contains most of the systems of interest in physics, biology, chemistry, ...

Now consider the embedding of system (4), obtained by adding to it the m following variables:

$$x_{N+j} = \prod_{k=1}^{N} x_k^{B_{jk}} \qquad (j = 1, \ldots, m).$$

Taking the time derivative of these new variables, we find the $(N+m)$-dimensional system:

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{p=1}^{N+m} M_{ip} x_p$$

which is of the form of the Lotka–Volterra (LV) system [11, 12]. These systems were first introduced to study the time evolution of interacting species. They appeared to have rich dynamical properties, and to show complex behaviours, like limit-cycles and even chaotic attractors. Even more: they seem to be the simplest form for a system of ODEs still showing this complex behaviour, since any further simplification (for example: the case of a degenerated matrix M) implies the integrability of the system.

In the above LV equations, i runs from 1 to $N+m$. The λ_i and the (square) matrix M are given by:

$$\begin{aligned} \lambda_i &= (\alpha_1, \ldots, \alpha_N, (B \circ \lambda)_1, \ldots, (B \circ \lambda)_m)^T \\ M &= \begin{pmatrix} 0^{(n\times n)} & A^{(n\times m)} \\ 0^{(m\times n)} & (B \circ A)^{(m\times m)} \end{pmatrix} \end{aligned} \tag{5}$$

where $(B \circ \lambda)_i = \sum_{k=1}^{N} B_{ik}\lambda_k$, and $(B \circ A)_{ij} = \sum_{k=1}^{N} B_{ik}A_{kj}$. In expression (5), the upper index gives the dimension of the matrices. The entries of $0^{(n\times n)}$ are all vanishing. Note that this LV system contains a closed subsystem, composed of the last m variables (the QMs). This subsystem contains all the information about the dynamics of the QM system. If, for instance, $N < m$, then it is possible to show that the QM system has first integrals that permits one to reduce its dimension from N to m. The use of the larger $(N+m)$-dimensional LV system is more natural, and practically, we do not have to worry about the inverse of the QM transformation [8, 9]: the original variables are just the first N variables of the LV system.

By adding one more variable to the system, which is set equal to one:

$$\dot{x}_{(N+m+1)} = 0 \qquad x_{(N+m+1)}(t = t_0) = 1$$

we can include the linear part in the matrix M:

$$\dot{x}_i = x_i \sum_{j=1}^{N+m+1} \acute{M}_{ij} x_j \tag{6}$$

with:

$$\acute{M} = \begin{pmatrix} M_{11} & M_{12} & \ldots & M_{1(n+m)} & \lambda_1 \\ M_{21} & M_{22} & \ldots & M_{2(n+m)} & \lambda_2 \\ \vdots & & \ddots & \vdots & \vdots \\ M_{(n+m)1} & M_{(n+m)2} & \ldots & M_{(n+m)(n+m)} & \lambda_{(n+m)} \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

2. Structure of the Poincaré transformation

Since we know that any QM system can be brought to the canonical LV form, we now focus on constructing the Poincaré series for LV systems (the generalization to QM system will be straightforward):

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{i=1}^{N} M_{ij} x_j \qquad (i = 1, \ldots, N) \tag{7}$$

where the linear part is explicitly written. It has been known, since the work of Carleman [13, 14], that a nonlinear system can be viewed as an infinite-dimensional linear system. This

can be realized by considering as new variables all the monomials one can build with products of positive integer powers of the x_i. Using the multi-index notation:

$$X_{\boldsymbol{m}} = x_1^{m_1} x_2^{m_2} \dots x_N^{m_N}$$

and derivating these new variables, we find:

$$\dot{X}_{\boldsymbol{m}} = (\boldsymbol{m}\cdot\boldsymbol{\lambda})X_{\boldsymbol{m}} + \sum_{p=1}^{N}\left(\sum_{l=1}^{N} m_l M_{lp}\right) X_{(\boldsymbol{m}+\boldsymbol{e}_p)}. \tag{8}$$

where $\boldsymbol{e}_p$ is a unit vector: $(\boldsymbol{e}_p)_s = \delta_{p,s}$, and $X_{\boldsymbol{m}+\boldsymbol{e}_p}$ is $X_{m_1,\dots,m_p+1,\dots,m_N}$. This infinite-dimensional linear system (8) is characterized by a triangular matrix R, which is given by

$$R_{\boldsymbol{m}\boldsymbol{p}} = (\boldsymbol{m}\cdot\boldsymbol{\lambda})\delta_{\boldsymbol{m},\boldsymbol{p}} + \sum_{k=1}^{N}\sum_{l=1}^{N} m_k M_{kl}\delta_{(\boldsymbol{m}+\boldsymbol{e}_l),\boldsymbol{p}}. \tag{9}$$

For an original system (7) that would be linear (i.e. the matrix M vanishes) system (8) would be purely diagonal. This implies that the Poincaré transformation on (7), for non-vanishing matrix M, but in absence of any resonance, corresponds to the diagonalization of the infinite-dimensional matrix defined by system (8), $R_{\boldsymbol{m}\boldsymbol{p}}$.

Consider now the operator L defined by the relation

$$L_{\boldsymbol{m}\boldsymbol{p}} = \delta_{\boldsymbol{m},\boldsymbol{p}} + \frac{\sum_{\boldsymbol{k}} R_{\boldsymbol{m}\boldsymbol{k}}(1-\delta_{\boldsymbol{m},\boldsymbol{k}})L_{\boldsymbol{k}\boldsymbol{p}}}{R}_{\boldsymbol{p}\boldsymbol{p}} - R_{\boldsymbol{m}\boldsymbol{m}} + \delta_{\boldsymbol{m}}, \boldsymbol{p} \tag{10}$$

where, once again, the indices are multiple (the sum over $\boldsymbol{k}$ is a multi-sum over the N indices k_i running from 0 to ∞, and $\delta_{\boldsymbol{m},\boldsymbol{p}}$ stands for $\delta_{m_1,p_1}\dots\delta_{m_N,p_N}$).

If $R_{\boldsymbol{m}\boldsymbol{p}}$ is triangular (that is: $R_{\boldsymbol{m}\boldsymbol{p}} = 0$ if there exists at least one integer k between 1 and N, such that $m_k > p_k$) and if $R_{\boldsymbol{m}\boldsymbol{m}} \neq R_{\boldsymbol{p}\boldsymbol{p}}$ (that is $\boldsymbol{m}\cdot\boldsymbol{\lambda} \neq \boldsymbol{p}\cdot\boldsymbol{\lambda}$) for all $\boldsymbol{m} \neq \boldsymbol{p}$, then relation (10) defines $L_{\boldsymbol{m}\boldsymbol{p}}$ without ambiguity. Indeed, the denominator never vanishes, and, for any finite $\boldsymbol{m}$ and $\boldsymbol{p}$, $L_{\boldsymbol{m}\boldsymbol{p}}$ is given by a finite sum of terms. The condition $R_{\boldsymbol{m}\boldsymbol{m}} \neq R_{\boldsymbol{p}\boldsymbol{p}}$ for $\boldsymbol{m} \neq \boldsymbol{p}$ implies that the relation $\boldsymbol{r}\cdot\boldsymbol{\lambda} = 0$, where $\boldsymbol{r}$ is a vector of positive or vanishing integers, is satisfied only if $r_k = 0$ for all k and restricts system (7) to the non-resonant case: it is easy to see that resonances of the kind $\boldsymbol{r}\cdot\boldsymbol{\lambda} = \lambda_i$ with $r_i = 0$ has no implication on the Poincaré series for a LV system, and more generally, for differential systems of the form $\dot{x}_i = \lambda_i x_1 + x_i f_i(x)$ with f_i analytic.

Given these conditions, we claim that L diagonalizes R. More precisely, in our case this means that, considering the inverse operator L^{-1} defined by

$$\sum_{\boldsymbol{k}} L^{-1}_{\boldsymbol{m}\boldsymbol{k}} L_{\boldsymbol{k}\boldsymbol{p}} = \sum_{\boldsymbol{k}} L_{\boldsymbol{m}\boldsymbol{k}} L^{-1}_{\boldsymbol{k}\boldsymbol{p}} = \delta_{\boldsymbol{m},\boldsymbol{p}} \tag{11}$$

we have:

$$\sum_{\boldsymbol{k},o} L^{-1}_{\boldsymbol{m}\boldsymbol{k}} R_{\boldsymbol{k}o} L_{o\boldsymbol{p}} = R_{\boldsymbol{m}\boldsymbol{p}}\delta_{\boldsymbol{m}.\boldsymbol{p}}. \tag{12}$$

The question of the existence of the inverse operator L^{-1} is obvious. This operator represents the inverse of the Poincaré transformation. The latter is a diffeomorphism. Hence, when it exists, so does its inverse. The analogue of the relation (10) for L^{-1} is

$$L^{-1}_{\boldsymbol{m}\boldsymbol{p}} = \delta_{\boldsymbol{m},\boldsymbol{p}} - \frac{\sum_{\boldsymbol{k}} L^{-1}_{\boldsymbol{m}\boldsymbol{k}} R_{\boldsymbol{k}\boldsymbol{p}}(1-\delta_{\boldsymbol{k},\boldsymbol{p}})}{R_{\boldsymbol{p}\boldsymbol{p}} - R_{\boldsymbol{m}\boldsymbol{m}} + \delta_{\boldsymbol{m},\boldsymbol{p}}}. \tag{13}$$

Let us now prove the proposition for the relation (10) (the demonstration for (13) goes along the same lines): multiplying both sides of (10) by $(R_{\boldsymbol{p}\boldsymbol{p}} - R_{\boldsymbol{m}\boldsymbol{m}} + \delta_{\boldsymbol{m},\boldsymbol{p}})$ one finds

$$\sum_{\boldsymbol{k}} R_{\boldsymbol{m}\boldsymbol{k}} L_{\boldsymbol{k}\boldsymbol{p}} = L_{\boldsymbol{m}\boldsymbol{p}} R_{\boldsymbol{p}\boldsymbol{p}} + \delta_{\boldsymbol{m},\boldsymbol{p}}(L_{\boldsymbol{m}\boldsymbol{p}} - 1) - \delta_{\boldsymbol{m},\boldsymbol{p}}(R_{\boldsymbol{p}\boldsymbol{p}} - R_{\boldsymbol{m}\boldsymbol{m}}). \tag{14}$$

We now show that the last two terms of (14) are vanishing. It is clearly the case for $\delta_{m,p}(R_{pp} - R_{mm})$. For $\delta_{m,p}(L_{mp} - 1)$ we demonstrate that, thanks to the fact that R_{mp} is triangular, so is L_{mp}, and all the elements L_{mm} are equal to one. To show this, consider the series obtained by iterating (10). This will be a series of powers of $R_{mp}(1 - \delta_{m,p})$ (with coefficients depending on $\boldsymbol{m}$ and $\boldsymbol{p}$). Now, the elements of the kth power of $R_{mp}(1 - \delta_{m,p})$ for which $p_i \leqslant m_i + k$ are all equal to zero. This implies that the only contribution to L_{mm} comes from the first term of the series, that is $\delta_{m,p}$. So, $L_{mm} = 1$, and $\delta_{m,p}(L_{mp} - 1) = 0$. Then multiplying (14) by L^{-1}_{om} and summing over $\boldsymbol{m}$, we find the announced result.

Taking this result back to the original LV system, we can build the Poincaré series for it. We insert the matrix (9) in (10) and, writing (10) for $\boldsymbol{m} = \boldsymbol{e}_i$, we find after some simple algebra:

$$x_i = y_i \sum_{n=0}^{\infty} \sum_{i_1,i_2,\ldots,i_n=1}^{N} P_{ii_1i_2\ldots i_n} y_{i_1} y_{i_2} \ldots y_{i_n} \tag{15}$$

where $i = 1, \ldots, N$ and with tensors P given by:

$$P_{ii_1i_2\ldots i_n} = \frac{M_{ii_1}(M_{ii_2} + M_{i_1i_2}) \ldots (M_{ii_n} + M_{i_1i_n} + \cdots + M_{i_{n-1}i_n})}{\lambda_{i_n}(\lambda_{i_{n-1}} + \lambda_{i_n}) \ldots (\lambda_{i_1} + \cdots + \lambda_{i_n})}. \tag{16}$$

The term corresponding to $n = 0$ is, by convention, set equal to 1.

The relation (13) can be used to compute the inverse of the Poincaré transformation:

$$y_i = x_i \sum_{n=0}^{\infty} (-1)^n \sum_{i_1,i_2,\ldots,i_n=1}^{N} I_{ii_1i_2\ldots i_n} x_{i_1} x_{i_2} \ldots x_{i_n} \tag{17}$$

with

$$I_{ii_1i_2\ldots i_n} = \frac{M_{ii_1}(M_{ii_2} + M_{i_1i_2}) \ldots (M_{ii_n} + M_{i_1i_n} + \cdots + M_{i_{n-1}i_n})}{\lambda_{i_1}(\lambda_{i_1} + \lambda_2) \ldots (\lambda_{i_1} + \cdots + \lambda_{i_n})}. \tag{18}$$

Note that the dependence on the matrix M is exactly the same as for the direct Poincaré transformation; the only difference is the order of the indices of the λ_i in the denominator.

Using the QM formalism, we can now build the analogues of the Poincaré transformation for a general QM system. We merely have to put the M given in (5), and to write (15) for the variables of the QM system. This gives

$$\begin{aligned} x_i &= y_i \sum_{s=0}^{\infty} \sum_{j_1,\ldots,j_s=1}^{m} P^{(QM)}_{ij_1\ldots j_s} \left(\prod_{k_1=1}^{N} y_{k_1}^{B_{j_1k_1}} \right) \cdots \left(\prod_{k_s=1}^{N} y_{k_s}^{B_{j_sk_s}} \right) \\ &= y_i \sum_{s=0}^{\infty} \sum_{j_1,\ldots,j_s=1}^{m} P^{(QM)}_{ij_1\ldots j_s} \prod_{k=1}^{N} y_k^{B_{j_1k} + B_{j_2k} + \cdots + B_{j_nk}}. \end{aligned} \tag{19}$$

Now, $P^{(QM)}_{ij_1\ldots j_s}$ is given by:

$$\frac{A_{ij_1}(A_{ij_2} + (B \circ A)_{j_1j_2}) \ldots (A_{ij_s} + (B \circ A)_{j_1j_s} + \cdots + (B \circ A)_{j_{(s-1)}j_s})}{(B \circ \lambda)_{j_s}((B \circ \lambda)_{j_{(s-1)}} + (B \circ \lambda)_{j_s}) \ldots ((B \circ \lambda)_{j_1} + \cdots + (B \circ \lambda)_{j_s})}. \tag{20}$$

The simple form of the LV system also allows to write explicitly the hierarchy (3), even for a resonant system; one has to substitute in the LV system the series

$$x_i = y_i \sum_{\substack{\boldsymbol{m} \\ |m|>0}} b_i(\boldsymbol{m}) y^{\boldsymbol{m}}.$$

The time derivatives of the y variables are

$$\dot{y}_i = \lambda_i y_i + y_i \sum_{\substack{\boldsymbol{m} \\ |m|>0}} c_i(\boldsymbol{m}) y^{\boldsymbol{m}}.$$

One then gets

$$\sum_{r,m} c_i(r)b_i(m)\ \ y^{r+m} + \sum_{r}(r\cdot\lambda)b_i(r)\ \ y^r + \sum_{r,m}\sum_{j=1}^{N} b_i(r)(r\cdot c(m))\ \ y^{r+m}$$
$$= \sum_{j=1}^{N} M_{ij} \sum_{r,m} b_i(m)b_j(r)y^{r+m+e_j}.$$

This equation implies the following infinite set of equations:

$$(r\cdot\lambda)b_i(m) = \sum_{j=1}^{N} M_{ij} \sum_{m_1+m_2=r-e_j} b_i(m_1)b_j(m_2)$$
$$- \sum_{m_1+m_2=r} b_i(m_1)[c_i(m_2) + (m_1\cdot c(m_2))]. \tag{21}$$

In the resonant case, the $c_i(m)$ can be chosen equal to zero as far as no resonance is encountered. When the lowest-order resonance occurs, say for $r = r_0$ with $|r_0| = n_0$, the left-hand side of the equation is vanishing. One then gets the equation

$$c_i(r_0) = \sum_{j=1}^{N} M_{ij} \sum_{m_1+m_2=r_0-e_j} b_i(m_1)b_j(m_2).$$

In the non-resonant case, one has:

$$(r\cdot\lambda)b_i(m) = \sum_{j=1}^{N} M_{ij} \sum_{m_1+m_2=r-e_j} b_i(m_1)b_j(m_2)$$

which is almost the same expression as the previous one for the normal form coefficient. Comparing this to the result obtained via the Carleman embedding, we are able to write

$$C_i(r_0) = \sum_{e_{i_1}+\cdots+e_{i_{n_0}}=r_0} \frac{M_{ii_1}(M_{ii_2} + M_{i_1i_2})\ldots(M_{ii_{n_0}} + \cdots + M_{i_{n_0-1}i_{n_0}})}{\lambda_{i_{n_0}}(\lambda_{i_{n_0-1}} + \lambda_{i_{n_0}})\ldots(\lambda_{i_1} + \cdots + \lambda_{i_{n_0-1}})}$$

which in fact reduces to

$$C_i(r_0) = \sum_{e_{i_1}+\cdots+e_{i_{n_0}}=r_0} \frac{M_{ii_1}M_{i_1i_2}(M_{i_1i_3} + M_{i_2i_3})\ldots(M_{i_1i_{n_0}} + \cdots + M_{i_{n_0-1}i_{n_0}})}{\lambda_{i_{n_0}}(\lambda_{i_{n_0-1}} + \lambda_{i_{n_0}})\ldots(\lambda_{i_1} + \cdots + \lambda_{i_{n_0-1}})}$$

The above results give compact expressions for the coefficients of the Poincaré series. They are, however, not easily found from the hierarchy (21), even in the non-resonant case, because it does not give the tensors $P_{ij_1\ldots j_s}$ as they are given in (16), but their symmetrized form, which are much longer to write. Furthermore, the similarity in the structure of these coefficients with the coefficients of the Taylor series obtained in [8] is striking.

3. Resonances in the QM context

The results presented in the previous sections were already briefly presented in [3]. We now present new developments about the linearizability character of non-analytical vector fields.

We thus derived the general structure of the Taylor and the Poincaré series for LV systems, and we showed that the QM formalism permits one to extend these results to any QM system. Consequently, it is necessary to extend the usual notion of resonance to systems that are not especially analytic. We do this by associating a resonance with a vanishing factor in the denominator of a coefficient (20). Restricting ourselves to analytic systems, we show that

vanishing denominators may appear even in the non-resonant case [2]. These resonances are called fictive resonances. They have no consequence, due to the analytical character of the vector field. Indeed, the factor that multiplies the fictive resonance is always vanishing. The extension of this notion to the more general QM context is straightforward, and, in fact, gives the conditions under which the resonant monomials do not appear in the normal form system.

3.1. Resonance

In the QM context, the usual resonance relation has to be modified in order to include all possible vanishing denominator in (20). This generalized resonance condition is:

$$\sum_{j=1}^{m} v_j \sum_{k=1}^{n} B_{jk}\lambda_k = \boldsymbol{v}\cdot\boldsymbol{B}\cdot\boldsymbol{\lambda} = 0 \tag{22}$$

with v_j a positive or vanishing integer (and $|\boldsymbol{v}| \geqslant 1$).

Let us precise that the resonance of an analytic system does not imply necessarily that there exists a $\boldsymbol{v}$ satisfying (22). To show this, let us consider an example of analytic resonant system:

$$\dot{x}_1 = -x_1 + \alpha x_1^2 x_2^2 + \beta x_1 x_2$$
$$\dot{x}_2 = x_2 + \gamma x_1 x_2^3 + \delta x_2^3.$$

This system is clearly resonant: $\boldsymbol{v}\cdot\boldsymbol{\lambda} = 0$ for $\boldsymbol{v} = (v, v)$. The QM matrices, and the vector $\boldsymbol{\lambda}$ are

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \qquad A = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & 0 & \delta \end{pmatrix} \qquad \text{and} \qquad \lambda = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence,

$$B \circ \lambda = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

and relation (22) is never satisfied.

When a situation like this arises, it means that the relation (3) corresponding to the resonance is trivially satisfied (it reduces to '$0 = 0$', that is $(\boldsymbol{m}\cdot\boldsymbol{\lambda} - \lambda_i)$ and $F_i(\boldsymbol{m})$ are vanishing in the same time). The resonance is compatible with the nonlinearity, in the sense that it is not an obstacle to the linearization (the corresponding coefficient of the Poincaré series is undetermined, and does not affect the coefficient of the lowest order monomials of the normal form system). On the other hand, we may also have a situation in which a relation (22) is satisfied, with a normal form system that is still linear. This will be discussed in the next section.

3.2. Fictive resonances

We start with a QM system:

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{N} x_k^{B_{jk}}$$

and we suppose it is analytic. This implies conditions on the matrices A and B: each monomial (including the factor x_i) has to be composed of integer positive powers of the variables. So, for a fixed index j between 1 and m, B_{jk} is a positive or vanishing integer number, except for

one index k (between 1 and N), for which we can have $B_{jk} = -1$; in this case (i.e., for j and k such that $B_{jk} = -1$) we have the condition on A: A_{ij} (with the same j than in B_{jk}) is vanishing for every index $i \neq k$. We also impose the condition that $\sum_{k=1}^{N} B_{jk} > 1$, in such a way that the linear part of the system is limited to the terms $\lambda_i x_i$.

In what follows, we consider that this system is not resonant, i.e. there is no N-dimensional vector $\boldsymbol{v}$, with integer positive (or vanishing) components v_k such that

$$\boldsymbol{v} \cdot \boldsymbol{\lambda} = \sum_{k=1}^{N} v_k \lambda_k = 0 \qquad \text{or} \qquad \lambda_i .$$

In this case, the general theory of normal form, introduced in the beginning of this paper, tell us that the system can be (at least formally) linearized, and that there will be no vanishing denominator in the Poincaré transformation. If we simply write the series (19), we obtain the Poincaré series for the QM system; but this time, it really is the Poincaré transformation since the system is analytic. Moreover, the absence of any resonance lets us think that no vanishing denominator will be encountered in the calculation. However, if one naively calculates the Poincaré series (19), one may still find vanishing denominators: these are called *fictive resonances*. When looking more carefully, one observes the following: the numerators corresponding to fictive resonances are vanishing too. Moreover, the corresponding monomials contain negative powers of the variables, and are thus not considered at all in the usual approach (i.e. using the recursion (21)). Let us now show this more explicitly.

A fictive resonance corresponds to a vanishing sum of $(B \circ \lambda)_j$. Let this particular vanishing combination be

$$0 = (B \circ \lambda)_{j_1} + (B \circ \lambda)_{j_2} + \cdots + (B \circ \lambda)_{j_s} = \boldsymbol{v} \cdot \boldsymbol{\lambda}.$$

In order to respect the non-resonant character of the original equations, the vector of integers $\boldsymbol{v}$ has to contain either one component which is smaller than or equal to -2, or at least two components equal to -1. In other words, *the sum of all the negative components of* $\boldsymbol{v}$ *has to be smaller than or equal to* -2.

The first consequence of this is, as announced, that the corresponding monomial contains negative powers of the variables. It is given by

$$\left(\prod_{k_1=1}^{N} y_{k_1}^{B_{j_1 k_1}} \right) \cdots \left(\prod_{k_s=1}^{N} y_{k_s}^{B_{j_s k_s}} \right) = \prod_{k=1}^{N} y_k^{B_{j_1 k} + B_{j_2 k} + \cdots + B_{j_s k}} = \prod_{k=1}^{N} y_k^{v_k}$$

and even with the factor y_i which multiplies this monomial in series (19), there will still be some variables with negative exponents.

If we push this analysis further we have to look at the consequences of the presence of a fictive resonance on the corresponding numerator. To do this, let us first treat a particular case, and consider a fictive resonance of the form

$$0 = (B \circ \lambda)_1 + (B \circ \lambda)_2 = \boldsymbol{v} \cdot \boldsymbol{\lambda}$$

with $v_1 = v_2 = -1$ and $v_k > -1$ for $k \neq 1, 2$. We can choose, without loss of generality, $B_{11} = B_{22} = -1$, and $B_{12} = B_{21} = 0$. This fictive resonance will appear, for the first time, in the term corresponding to $s = 2$ in expressions (19) and (20):

$$y_i \sum_{j_1, j_2 = 1}^{m} \frac{A_{ij_1}(A_{ij_2} + (B \circ A)_{j_1 j_2})}{(B \circ \lambda)_{j_2}[(B \circ \lambda)_{j_1} + (B \circ \lambda)_{j_2}]} \prod_{k=1}^{N} y_{k_1}^{B_{j_1 k} + B_{j_2 k}}$$

when, either $j_1 = 1$ and $j_2 = 2$, or $j_1 = 2$ and $j_2 = 1$. In the first case, we have as the numerator

$$A_{i1}(A_{i2} + (B \circ A)_{12}) = A_{i1}A_{i2} + A_{i1}(B \circ A)_{12}.$$

We know that A_{i1} is vanishing, except for $i = 1$; we also know that $A_{i2} = 0$, except for $i = 2$. So, the products $A_{i1}A_{i2}$ is always vanishing, whatever i is. Moreover, $(B \circ A)_{12}$ is

$$\sum_{k=1}^{N} B_{1k}A_{k2} = B_{12}A_{22} = 0$$

(the first equality is a consequence of $A_{k2} \sim \delta_{k,2}$, the second is due to $B_{12} = 0$) and so the whole numerator is vanishing. The same holds for $j_1 = 2$, $j_2 = 1$.

Let us now take a look at the next order ($s = 3$ in (19) and (20)); the coefficient is:

$$\frac{A_{ij_1}(A_{ij_2} + (B \circ A)_{j_1 j_2})(A_{ij_3} + (B \circ A)_{j_1 j_3} + (B \circ A)_{j_2 j_3})}{(B \circ \lambda)_{j_3}[(B \circ \lambda)_{j_2} + (B \circ \lambda)_{j_3}][(B \circ \lambda)_{j_1} + (B \circ \lambda)_{j_2} + (B \circ \lambda)_{j_3}]}. \tag{23}$$

Problems will arise either when $j_2 = 1$ and $j_3 = 2$, or when $j_2 = 2$ and $j_3 = 1$. In the first situation, the factor of the fictive resonance is

$$\frac{A_{ij_1}(A_{i1} + (B \circ A)_{j_1 1})(A_{i2} + (B \circ A)_{j_1 2})}{(B \circ \lambda)_2 (B \circ \lambda)_{j_1}}.$$

where we took into account the fact that $(B \circ A)_{12} = 0$. If, for example, $i = 1$, we have

$$\frac{A_{1j_1}(A_{11} + (B \circ A)_{j_1 1})(B \circ A)_{j_1 2}}{(B \circ \lambda)_2 (B \circ \lambda)_{j_1}}$$

and this quantity is not vanishing, so we have to take into account the symmetric contribution corresponding to $j_2 = 2$ and $j_3 = 1$ in (23):

$$\frac{A_{ij_1}(A_{i2} + (B \circ A)_{j_1 2})(A_{i1} + (B \circ A)_{j_1 1})}{(B \circ \lambda)_1 (B \circ \lambda)_{j_1}}.$$

Putting these two contributions to the factor together, we find

$$\frac{A_{ij_1}(A_{i1} + (B \circ A)_{j_1 1})(A_{i2} + (B \circ A)_{j_1 2})}{(B \circ \lambda)_{j_1}} \left(\frac{1}{(B \circ \lambda)_1} + \frac{1}{(B \circ \lambda)_2} \right)$$

and since $(B \circ \lambda)_1 + (B \circ \lambda)_2 = 0$, this is vanishing.

The main difference between the cases $n = 2$ and $n = 3$, is the following: in the first situation, each contribution to the factor of the fictive resonance vanishes separately. In the second situation, the two symmetric contributions kill each other. This is in fact a generic feature: the fictive resonance disappears when one symmetrizes the tensor $P^{(QM)}_{ij_1 \ldots j_s}$ in its indices $j_1 \ldots j_s$. This is due to the way the Poincaré series is written: a given monomial $\boldsymbol{y}^{\boldsymbol{m}}$ of order s ($|\boldsymbol{m}| = s$) appears as soon as the ordered sequence of indices $j_1 \ldots j_s$ is a permutation of the set of indices that contains m_1 times 1, m_2 times 2,..., m_N times N. Thus, to find the actual coefficient of $\boldsymbol{y}^{\boldsymbol{m}}$, we have to sum the $P^{(QM)}_{ij_1 \ldots j_s}$ over all the set of indices $j_1 \ldots j_s$ that are such that

$$\boldsymbol{e}_{j_1} + \boldsymbol{e}_{j_2} + \cdots + \boldsymbol{e}_{j_s} = \boldsymbol{m}$$

where $\boldsymbol{e}_{j_\alpha}$ is the unit vector in the direction j_α.

From a theoretical point of view, this may seem of little interest (from a practical point of view, however, this phenomenon has to be taken into account). The following remark shows that it is not the case: in the QM context, there is no fundamental difference between an analytic system that contains a fictive resonance, and a system with a resonance, but whose normal form is linear (this is the situation in which the relation (3) is satisfied because both $(\boldsymbol{m} \cdot \boldsymbol{\lambda} - \lambda_i)$ and $F_i(\boldsymbol{m})$ are vanishing). In the two situations, particular restrictions on the matrix M of the LV system make the coefficient of the resonance vanish. So we can find conditions on the parameters of the vector field such that a given resonance is not an obstacle to linearization.

To specify the main idea a little more (but without entering details), let us consider a resonance:

$$\boldsymbol{m}\cdot\boldsymbol{\lambda}=0 \qquad |\boldsymbol{m}|=r.$$

This resonance can be written

$$\lambda_{\alpha_1}+\lambda_{\alpha_2}+\ldots\lambda_{\alpha_r}=0,$$

with $(\alpha_1,\alpha_2,\ldots,\alpha_r)$ an ordered sequence such that:

$$(\alpha_1,\alpha_2,\ldots,\alpha_r)\in \text{permutation of}\quad (\underbrace{1,\ldots,1}_{m_1\times},\underbrace{2,\ldots,2}_{m_2\times},\ldots,\underbrace{N,\ldots,N}_{m_N\times}).$$

The resonance appears for the first time at the order $n=r$, and the condition $F_i(\boldsymbol{m})=0$ in (3) here becomes

$$\sum_{\text{perm}(\alpha_1,\ldots,\alpha_N)} P_{i\alpha_1\ldots\alpha_N}\times[\lambda_{\alpha_N}(\lambda_{\alpha_{N-1}}+\lambda_{\alpha_N})\ldots(\lambda_{\alpha_2}+\cdots+\lambda_{\alpha_N})]^{-1}=0$$

for all i. The sum has to be understood as the sum over all the different permutations of the fixed indices $(\alpha_1\alpha_2\ldots\alpha_r)$.

At the higher orders, what happens depends on the degree of the resonance (i.e. the number of free parameters in $\boldsymbol{m}$).

4. Generalized resonances

The method of this section is based on changes of variables that modify the linear part of the vector field. Doing so, we will be able to write a generalized resonance condition. This is, of course, impossible to do with the usual theory of normal forms, since such a change of variables will in general make negative powers of the variables appear. In the QM context, this can be allowed. However, we have to strongly reduce the class of transformations. Indeed, a simple linear change of variable, for instance, will map a QM system that contains negative or non-integer powers of the variables, onto a system that is no longer a QM system. We will in fact use two kinds of transformation; let us introduce the first, the so-called QM changes of variables. As we will show, these transformations have the property that they do not change the resonant or non-resonant character of the vector field. These changes of variables are of the form

$$u_i=\prod_{k=1}^{N}x_k^{C_{ik}} \qquad i=1,\ldots,N$$

where C is an invertible squared matrix. This gives the new QM system

$$\dot{u}_i=\lambda'_i u_i+u_i\sum_{j=1}^{m}A'_{ij}\prod_{k=1}^{N}u_k^{B'_{jk}}$$

with matrices A', B' and vector λ' are given by

$$A'=C\circ A$$
$$B'=B\circ C^{-1}$$
$$\lambda'=C\circ\lambda.$$

The products $(B'\circ A')$ and $(B'\circ\lambda')$ are unchanged, hence the resonance condition is invariant.

The second kind of transformations that preserve the QM character of the system are the new time transformations (NTT). In contrast with the former, these transformations will

modify the resonance condition and yield a generalized condition on the matrix $(B \circ A)$. A NTT reads

$$\mathrm{d}t = \left(\prod_{k=1}^{N} x_k^{\beta_k} \right) \mathrm{d}t'$$

where the β_k are arbitrary reals, and where x_i is expressed in term of the new variable t'. Inserting it in the QM system, we find

$$\frac{\mathrm{d}x_i}{\mathrm{d}t'} = \lambda_i x_i \left(\prod_{k=1}^{N} x_k^{\beta_k} \right) + x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{N} x_k^{B_{jk}+\beta_k}.$$

The most important difference with a simple QM transformation and, in fact, the reason for which the resonance condition will be modified, is that the action of a NTT mixes the linear and the nonlinear terms. It is thus natural to modify the notation in order to include the linear part in the matrices A and B. Let us rewrite the QM system (4)

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{N} x_k^{B_{jk}}. \tag{24}$$

where the matrix B is now

$$B = \begin{pmatrix} B_{11} & \dots & B_{1N} \\ B_{21} & \dots & B_{2N} \\ \vdots & & \vdots \\ B_{(m-1)1} & \dots & B_{(m-1)N} \\ 0 & \dots & 0 \end{pmatrix}.$$

The coefficients of the linear terms are here given by the last column of A, and, in general, by the lth column of A when all the elements of the lth line of B are vanishing. With this notation, the λ_i that enters the resonance condition (22) are given by the A_{il} for that fixed value of l. The NTT do not modify A, but B becomes:

$$B' = \begin{pmatrix} B_{11} + \beta_1 & \dots & B_{1n} + \beta_N \\ B_{21} + \beta_1 & \dots & B_{2n} + \beta_N \\ \vdots & & \vdots \\ B_{(m-1)1} + \beta_1 & \dots & B_{(m-1)N} + \beta_N \\ \beta_1 & \dots & \beta_N \end{pmatrix}. \tag{25}$$

In the general case, there is no vanishing line in this matrix, hence, no linear term in the new system. But if we choose $\beta_k = -B_{lk}$ for all k and a fixed l, then the lth line of the new matrix B' of expression (25) is vanishing. The new spectrum is given by $\lambda_k = A_{kl}$ (with the same fixed l). The matrix B appearing in the resonance relation (22) is not the matrix B' (25). This one has one line too much: the one corresponding to the linear terms, i.e. the line of zeros. The resonance condition for the system after the NTT is thus:

$$\sum_{k=1}^{N} \sum_{\substack{j=1 \\ j \neq l}}^{m} v_j (B_{jk} - B_{lk}) A_{kl} = 0$$

with $\sum_{j=1, j\neq l}^{m} v_j \geqslant 1$. It can also be written:

$$\sum_{j=1}^{m} v_j (B \circ A)_{jl} = |\boldsymbol{v}| (B \circ A)_{ll} \qquad \text{with} \quad v_l = 0 \quad \text{and} \quad |\boldsymbol{v}| = \sum_{j=1}^{m} v_j > 0. \tag{26}$$

This condition is obviously more general than the condition (22), and clearly reduces to it for a particular choice of l.

Let us now try to make the meaning of (26) more precise. If there is at least one index l such that the condition (26) is never satisfied, then the system can be reduced to a non-resonant system. Suppose we find such an l, then perform the NTT:

$$\mathrm{d}t = \mathrm{d}t' \left(\prod_{k=1}^{N} x_k^{-B_{lk}} \right)$$

where x_k denotes now $x_k(t')$. Clearly, the system so obtained is not resonant (note that in this mapping, an initially analytic system becomes non-analytic). We can thus use the QM-Poincaré transformation to linearize it. This will give

$$\frac{\mathrm{d}y_i}{\mathrm{d}t'} = A_{il} y_i. \tag{27}$$

Let us now show the meaning of this for the original resonant system. We have the following scheme of transformation:

$$\begin{array}{ccc} & \overset{P'}{\Longrightarrow} & \\ \text{original resonant system } (t) & & \text{unknown system } (t) \\ (NTT) \Downarrow & & \Uparrow (NTT)^{-1} \\ \text{non-resonant system } (t') & \underset{P}{\Longrightarrow} & \text{linear system } (t') \end{array}$$

Where NTT and $(NTT)^{-1}$ respectively denote the new time transformation and its inverse, P is the generalized Poincaré transformation performed on the system obtained after NTT, and P' is given by

$$P' = (NTT)^{-1} \circ P \circ NTT.$$

The local character of P implies that P and NTT commute, and thus $P' = P$. This means that we can perform P directly on the original system. The unknown system of the scheme, is then given by $(NTT)^{-1}$ acting on the linear system, hence by

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = y_i A_{il} f(y).$$

The function $f(y)$ is just the quasi-monomial $\prod_{k=1}^{N} x_k^{B_{lk}}$ expressed in terms of the y variables.

From a geometrical point of view, the phase space of this system is strictly the same as the one of the system (27), since the NTT is just a reparametrization of time. Hence, we have reached the goal of the usual normal form theory: sketch the phase-portrait of the system. There is, however, a question that has no answer at this stage: the usual normal form approach is local: it only concerns a neighbourhood of the fixed point under study. What about the NTT approach?

In order to simplify the discussion, let us focus on the (homogeneous) LV embedding, instead of the QM system. Then, if the system before the NTT reads

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i \sum_{j=1}^{N} M_{ij} x_j$$

(where, eventually, the entries of one of the lines of the matrix M are all vanishing) the only NTT that will preserve a linear part, reads

$$\mathrm{d}t = \mathrm{d}t' \, x_l^{-1}$$

with l integer between 1 and N. With no loss of generality, we may choose $l = N$. The system then becomes

$$\frac{\mathrm{d}x_i}{\mathrm{d}t'} = x_i \sum_{j=1}^{N} M_{ij} x_j x_N^{-1}.$$

Let us now perform the LV embedding on this system: we have to add the variables

$$x_{N+i} = x_i x_N^{-1}.$$

for $i < N$. We then have

$$\frac{\mathrm{d}x_i}{\mathrm{d}t'} = M_{iN} x_i + x_i \sum_{j=1}^{N-1} M_{ij} x_{N+j}$$

for i varying from 1 to N, and

$$\frac{\mathrm{d}x_{N+i}}{\mathrm{d}t'} = x_{N+i}(M_{iN} - M_{NN}) + x_{N+i} \sum_{j=1}^{N-1} (M_{ij} - M_{Nj}) x_{N+j}$$

for $i = 1, \ldots, N-1$. We now have a new LV system on which we may perform the Poincaré transformation (15). The structure of the matrix defined by this new $(2N-1)$-dimensional system has the consequence that the only monomial appearing in the Poincaré transformation, are monomial of the variables $x_{N+1}, \ldots x_{2N-1}$. Hence, viewed as a transformation on the original system, these monomials are monomials in $\frac{x_1}{x_N}, \ldots, \frac{x_{N-1}}{x_N}$. As a consequence of that, the point of phase space around which the series is built is: $(0, 0, \ldots, 0, \infty)$.

As claimed, we have thus obtained a method to approximate the trajectories far from the fixed point under study. Moreover, the NTT can be performed with the other variables, providing each time a series that is valid in a different sector.

The situation is less clear for a QM system (or even for an analytic system), since the LV system from which we started, is already an embedding of a previous system. This means that the origin in the phase space of the LV system may already correspond to a point at infinity for the QM system. In this context, no general considerations can be made, except that the transformation can be, once again, explicitly written, using the formulae (20) and (19) with different parameters; starting with the system (24), we have:

$$x_i = y_i \sum_{s=0}^{\infty} \sum_{j_1, \ldots, j_s} \tilde{P}^{(QM)}_{i j_1 \ldots j_s} \prod_{k=1}^{N} y_k^{B_{j_1 k} + B_{j_2 k} + \cdots + B_{j_s k} - s \cdot B_{lk}}.$$

The second summation is taken over all the indices $j_1, \ldots, j_s$, each of them running from 1 to m, but without taking the value l. The $\tilde{P}^{(QM)}_{i j_1 \ldots j_s}$ are:

$$\tilde{P}^{(QM)}_{i j_1 \ldots j_s} = \frac{A_{i j_1}(A_{i j_2} + \tilde{M}_{j_1 j_2}) \ldots (A_{i j_s} + \tilde{M}_{j_1 j_s} + \cdots + \tilde{M}_{j_{s-1} j_s})}{\tilde{M}_{j_s l}(\tilde{M}_{j_{s-1} l} + \tilde{M}_{j_s l}) \ldots (\tilde{M}_{j_1 l} + \cdots + \tilde{M}_{j_s l})}$$

where $\tilde{M}_{ij} = \sum_k (B_{ik} - B_{lk}) A_{kj}$. We can check that these expressions become (19) and (20) when $B_{lk} = 0$ for all k, and $A_{kl} = \lambda_l$.

We also have to specify that, when integrating the NTT relation, one may find a relation between t' and t such that t' becomes infinite for a finite t.

In order to illustrate the method, we will now treat an example.

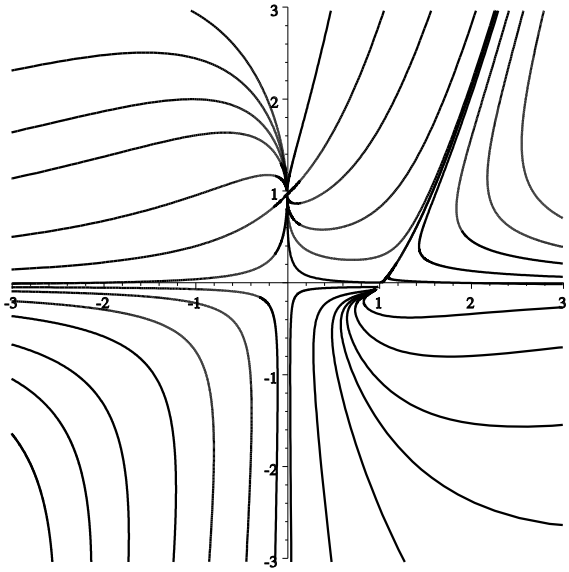

Figure 1. Phase portrait in the (n_1, n_2) plane.

4.1. Example

Let us study the following two-dimensional system:

$$\begin{aligned} \dot{x}_1 &= x_1 + x_1(-x_1 + x_2) \\ \dot{x}_2 &= -x_2 + x_2(x_1 + x_2) \end{aligned} \tag{28}$$

which is already of the LV type. This system possess three fixed points: $(0, 0)$, $(1, 0)$ and $(0, 1)$. The phase portrait in the neighbourhood of the origin is drawn in figure 1. Moreover, the fixed point under study, $(0, 0)$, is resonant.

The Poincaré transformation can be obtained by using (15), letting all the undetermined coefficients of (16) vanish. This gives, up to order three:

$$\begin{aligned} x_1 &= y_1(1 - y_1 - y_2 + y_1^2 + y_2^2) + \mathrm{o}(y^4) \\ x_2 &= y_2(1 + y_1 - y_2 + y_2^2) + \mathrm{o}(y^4). \end{aligned}$$

We now have to be careful in order to compute the normal form of the system. Indeed, the inverse Poincaré series obtained by using (17) and (18) (again letting the undetermined coefficients vanish) is not the inverse of the direct Poincaré transformation obtained above. We thus have to compute the inverse series by hand. This allows us to compute the normal form system, which reads:

$$\begin{aligned} \dot{y}_1 &= y_1 + 2y_1^2 y_2 + \mathrm{o}(y^4) \\ \dot{y}_2 &= -y_2 + \mathrm{o}(y^4). \end{aligned}$$

Figure 2 is a plot of the (x_1, x_2) plane. As expected, the normal form approximation (in grey) is close to the exact solution (in black) in the neighbourhood of $(0, 0)$. Higher-order approximations are expected to give better and better approximations. However, since the Poincaré series usually has a finite radius of convergence, nothing good can be obtained outside a certain neighbourhood of the origin. The calculation of the radius of convergence of the Poincaré series is not the object of this paper, thus we will not study this question here.

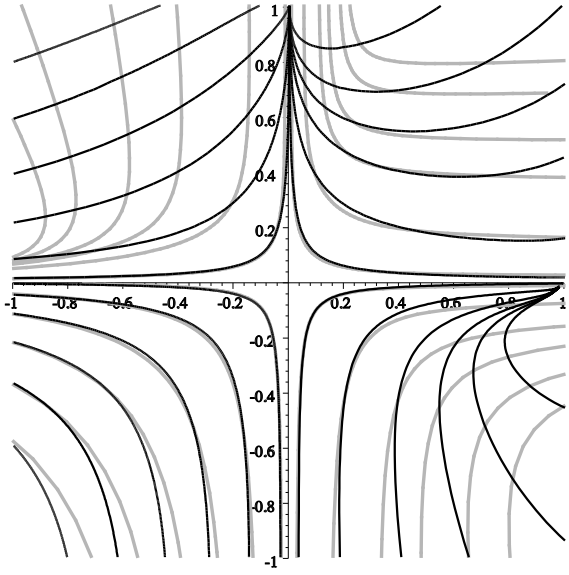

Figure 2. Normal form trajectories (in grey) versus exact trajectories (in black) around the fixed point.

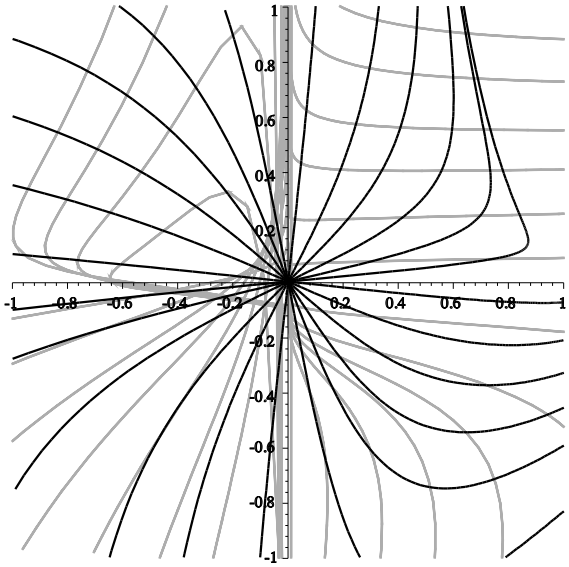

Figure 3. Comparison between the exact solutions (in black) and the normal form approximation (in grey) around the point $(\infty, 0)$.

Figure 3 shows the normal form approximation (in grey) in coordinates which are $(1/x_1, x_2)$. Clearly, the approximation fails.

Let us now change the time parametrization as follows:

$$\mathrm{d}t = \mathrm{d}t' \, x_1^{-1}.$$

This gives the system

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t'} &= -x_1 + x_1(x_1^{-1} + x_1^{-1}x_2)\\ \frac{\mathrm{d}x_2}{\mathrm{d}t'} &= x_2 + x_2(-x_1^{-1} + x_1^{-1}x_2).\end{aligned} \tag{29}$$

The LV embedding is obtained by adding the two following variables to this system:

$$\begin{aligned}x_3 &= x_1^{-1}\\ x_4 &= x_1^{-1}x_2\end{aligned}$$

with the use of which, we can write the four-dimensional LV system

$$\begin{aligned}x_1' &= -x_1 + x_1(x_3 + x_4)\\ x_2' &= x_2 + x_2(-x_3 + x_4)\\ x_3' &= x_3 + x_3(-x_3 - x_4)\\ x_4' &= 2x_4 + x_4(-2x_3)\end{aligned}$$

where the notation $x_1', x_2', \ldots$ denotes the derivative with respect to t'. The system seems to be resonant, but it is not the case, since the nonlinear terms are such that no coefficient is divergent in the Poincaré series (the system (29) is not resonant in the sense defined for the QM context). In fact, the only variables that appear in the Poincaré series are z_3 and z_4, which are in fact z_1^{-1} and $z_1^{-1}z_2$, where z_i denote the new Poincaré variables. The Poincaré series for the new four-dimensional system reads

$$\begin{aligned}x_1 &= z_1(1 + z_3 + \tfrac{1}{2}z_4 - \tfrac{1}{3}z_3z_4) + \mathrm{o}(z^4)\\ x_2 &= z_2(1 - z_3 + \tfrac{1}{2}z_4 + z_3^2 + \tfrac{1}{8}z_4^2 - z_3z_4) + \mathrm{o}(z^4)\\ x_3 &= z_3(1 - z_3 + \tfrac{1}{2}z_4 + z_3^2 + \tfrac{1}{8}z_4^2 + 4/3z_3z_4) + \mathrm{o}(z^4)\\ x_4 &= z_4(1 - 2z_3 + 3z_3^2 + \tfrac{1}{3}z_3z_4) + \mathrm{o}(z^4).\end{aligned}$$

Using the expression of z_3 and z_4 in terms of z_1 and z_2, one gets for x_1 and x_2

$$x_1 = z_1\left(1 + \frac{1}{z_1} + \frac{z_2}{2z_1} - \frac{z_2}{3z_1^2}\right) + \mathrm{o}\left(\frac{{z_2}^2}{z_1}\right)$$

$$x_2 = z_2\left(1 - \frac{1}{z_1} + \frac{z_2}{2z_1} + \frac{1}{z_1^2} + \frac{z_2^2}{8z_1^2} - \frac{z_2}{z_1^2}\right) + \mathrm{o}\left(\frac{{z_2}^2}{z_1}\right).$$

It is clear that the terms that we neglected above are important when x_1 gets close to zero. Hence, the approximation fails there. However, for x_2 close to zero and x_1 close to ∞, the approximation is good. Figure 4 shows, in the $(1/x1, x2)$ plane, the exact solution (in black) versus the above approximation (in grey). This is to be compared to the normal form result in the same domain (figure 3).

Systems with two quasi monomials

We now focus on exact properties related to generalized resonances (GR).

With the GR relation, we can show that a system that contains two monomials can either be brought to a non-resonant system or is simply integrable. For such systems, B has two lines and A two columns, hence $(B \circ A)$ is a (2×2) square matrix.

If only one (among the two possible) GR condition is satisfied, the system can be written as a non-resonant system. Otherwise, the two GR conditions are simultaneously satisfied. Let us write the GR condition for $l = 1$:

$$v_2(B \circ A)_{21} = v_2(B \circ A)_{11}$$

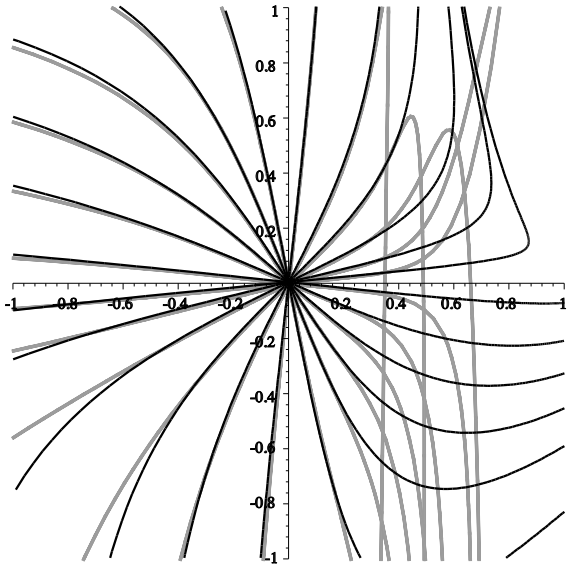

Figure 4. Comparison between the exct solutions (in black) and the normal form approximation obtained after new time transformation (in grey) around the point $(\infty, 0)$.

with a non-vanishing v_2. So we have $(B \circ A)_{21} = (B \circ A)_{11}$. The GR relation for $l = 2$ gives $(B \circ A)_{12} = (B \circ A)_{22}$. The matrix $(B \circ A)$ of the LV system is degenerated, and it reduces to one equation.

Non-resonant perturbations

We now show that a system such that all its GR conditions are satisfied can always be modified by adding one monomial to each equation, in such a way that it becomes reducible as shown above (i.e. to a system with one monomial).

Consider the N-dimensional QM system, with m monomials and such that there exists, for any index l between 1 and m, a vector $\boldsymbol{v}$ of integers that are positive or vanishing (with $v_l = 0$ for some l, and $|\boldsymbol{v}| \geqslant 1$), and satisfying

$$\sum_{j=1}^{m} v_j (B \circ A)_{jl} = |\boldsymbol{v}| (B \circ A)_{ll}.$$

Now let us add a monomial that could be a perturbation to the system, with free coefficients and exponents:

$$\dot{x}_i = x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} + x_i \alpha_i \prod_{k=1}^{n} x_k^{\beta_k}$$

and try to find values of the α_i and the β_k in such a way that a GR condition is never satisfied for this system.

The new QM matrices $\acute{A}$ and $\acute{B}$ read, in terms of the matrices A and B of the original

system:

$$\acute{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1m} & \alpha_1 \\ A_{21} & A_{22} & \dots & A_{2m} & \alpha_2 \\ \vdots & \vdots & & \vdots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nm} & \alpha_n \end{pmatrix}$$

and:

$$\acute{B} = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & \dots & B_{mn} \\ \beta_1 & \beta_2 & \dots & \beta_n \end{pmatrix}.$$

The product $(\acute{B} \circ \acute{A})$ is

$$(\acute{B} \circ \acute{A}) = \begin{pmatrix} (B \circ A)^{(m\times m)} & (B \circ \alpha)^{(m\times 1)} \\ (\beta \circ A)^{(1\times m)} & (\beta \circ \alpha)^{(1\times 1)} \end{pmatrix}.$$

Hence, the GR conditions are the following; first for l between 1 and m, we have

$$\sum_{j=1}^{m} v_j (B \circ A)_{jl} + v_{(m+1)} (B \circ \beta)_l = |\boldsymbol{v}| (B \circ A)_{ll}.$$

Now $\boldsymbol{v}$ is a vector with $(m + 1)$ components. If we choose, for the m first components of $\boldsymbol{v}$, the vector that satisfied the GR condition for the initial system, we are left with

$$v_{(m+1)} (B \circ \beta)_l = v_{(m+1)} (B \circ A)_{ll}$$

which is satisfied for $v_{m+1} = 0$, a choice which is compatible with the restrictions on the vector $\boldsymbol{v}$. Thus, for the m first columns of $(\acute{B} \circ \acute{A})$, it is not possible to fulfil the requirement that the system is not resonant.

Now let us look at the last column of $(\acute{B} \circ \acute{A})$; we have

$$\sum_{j=1}^{m} v_j (B \circ \alpha)_j = |\boldsymbol{v}| (\beta \circ \alpha)$$

or:

$$\sum_{j=1}^{m} v_j \left[\sum_{k=1}^{n} (B_{jk} - \beta_k) \alpha_k \right] = 0.$$

If we now choose $\beta_k > |B_{jk}|$ (here $|\cdot|$ denotes the absolute value) for all j and all k, and $\alpha_k > 0$ for all k, the relation will never be satisfied with an acceptable $\boldsymbol{v}$. Hence the system is linearizable. Note that we can choose $\alpha_k = \epsilon$ and $\beta_k = \beta$, with β as large as we want, and ϵ as small as we want, as in the case of a perturbation: such a perturbation would make a system integrable which originally was not.

5. Conclusions

In this paper, we presented new developments, among which the most interesting one is, without doubt, the generalization of resonances. Even if the generalization of linearizing transformations to more general mappings than simple Taylor series has been envisaged [15], to the best of our knowledge, the reduction scheme of a system to something else than its linear part is completely new. This new type of reduction opens interesting perspectives in the theory of integrability, and in the construction of invariants manifolds.

The explicit form of the generalized Poincaré series can be used to build computer algebra algorithms. This will be the subject of a further publication, in which we will develop a graphical tree method to perform resummations of the series. This formalism can also be used to optimize the recursion relations used to calculate the coefficients. The trees also permit one to unify the Taylor series in powers of the time, the Poincaré series and another series previously discussed in [3], which is a regular perturbation series. Numerical examples already indicate that the resummations of the series permit to reach a given accuracy with less calculations than in usual numerical integration methods. Furthermore, for regular perturbation series, it generates results close to singular perturbation methods.

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